International Baccalaureate

Mathematics Extended Essay

Paths of Shortest Distance on Earth

Research Question: How do geodesics, calculated using the Euler-Lagrange equation and the metric tensor, allow for calculations of the shortest distance between two points on a sphere?

Word Count: 3985

Session: May 2025

Contents

1 Introduction **1**

1.1 Geodesics 1

1.2 Defining the Methodologies 3

2 Prerequisites **5**

2.1 Multivariable Calculus 5

2.1.1 Multivariable Chain Rule 5

2.1.2 Leibniz's Integral Rule 6

2.2 Variational Calculus 7

2.3 Euler-Lagrange Equation 10

3 Geodesics **12**

3.1 Arc Length Integral 12

3.2 Deriving the Geodesic Equations 14

3.3 Distances on a Surface 15

3.4 Geodesic Equations for a Sphere 19

3.5 Determining the Constant c_2 25

3.6 Calculating Distances on the Earth 29

4 Conclusion **32**

5 References **33**

1 Introduction

1.1 Geodesics

One of the greatest advents of the age of exploration was Gerardus Mercator's invention of the Mercator Projection, first employed in his map titled *Nova et Aucta Orbis Terrae Descriptio ad Usum Navigantium Emendata Accommodata*[1].

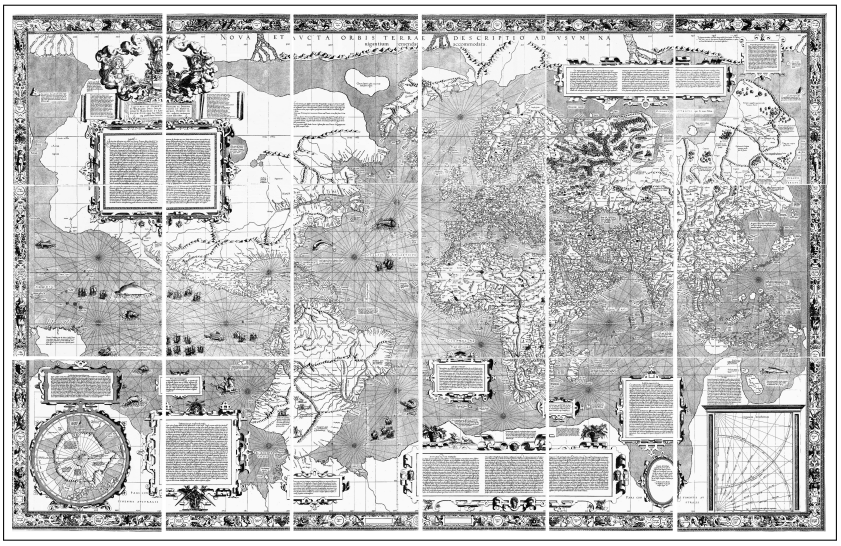

Figure 1: The first of many world maps to use the Mercator Projection.

Using a table of rhumbs, a precomputed set of values relating latitude changes to distances travelled along compass bearings, he developed a map in which straight lines corresponded to a line of constant bearing relative to true north, more formally, paths of constant azimuth (horizontal angle of distance). Unfortunately, this resulted in severe and unavoidable distortions.

Our cartesian plane is built upon the five postulates established by Euclid in his mathematical treatise *Elements*[2], c. 300 BC. However, when dealing with the Earth's surface, Euclidean assumptions no longer hold. In particular, the fifth postulate, which states that parallel lines never intersect, is violated on curved surfaces such as spheres or hyperbolic spaces. Notice the following plane trajectory:

[1] "Mercator, World Map (1569)." Cabinet, `www.cabinet.ox.ac.uk/mercator-worldmap-1569-1`
[2] Fitzpatrick, Richard. Euclid's Elements. Lulu.com, 2007.

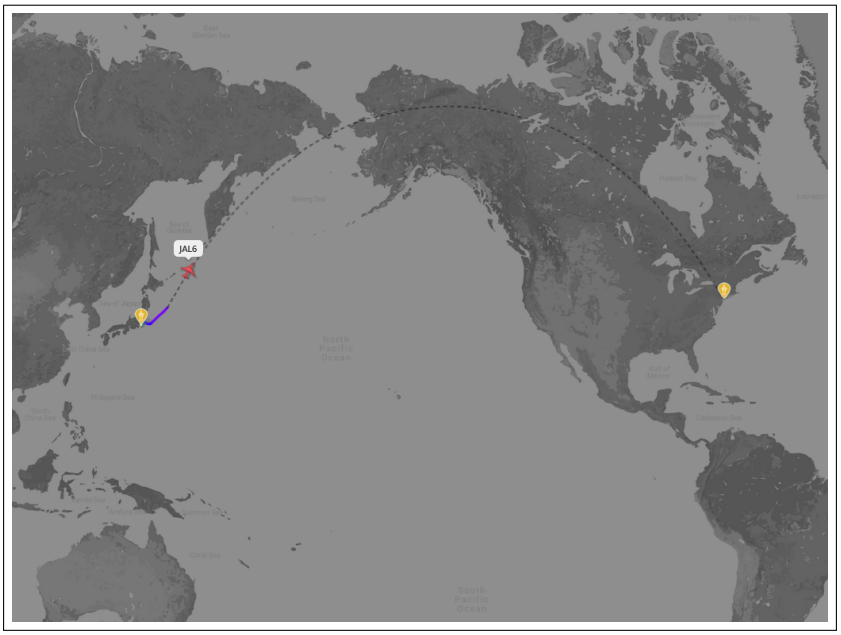

Figure 2: The projected flight path of "JL5" from Tokyo to New York [flightradar24]

At first glance, the flight path appears to deviate from the expected straight line. This perception arises because the path is viewed on a flat projection of the spherical Earth. In reality, however, the plane is following a straight line, or geodesic, along the globe's surface. A geodesic, etymologically derived from the field of Geodesy, which involves the study of the Earth's measurements, is a generalization of the concept of a "straight line" to curved surfaces and manifolds, such as the surface of the Earth. Locally, **a geodesic represents a length-minimizing curve**[3], and traveling along it would be following the most direct route possible across a curved surface.

From establishing flight paths to solving field equations, geodesics surprisingly plays a central role in understanding the world and universe around us. The aim of this extended essay is to use tools of differential geometry to formulate equations for geodesics passing through points on the surface of a sphere, the Earth, and use it to find paths of shortest distance. The research question is thus: *How do geodesics, calculated using the Euler-Lagrange equation and the Metric Tensor, allow for calculations of the shortest distance between two points on a Sphere?*

[3] "Geodesic." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource, 16 Apr. 2002, `mathworld.wolfram.com/Geodesic.html`.

1.2 Defining the Methodologies

We have two approaches to calculate geodesics along surfaces. Firstly, intrinsic geometric methods can be used, which calculates distances with respect to the curvature of the surface. Taking a rope from one location on the Earth to another would be an intrinsic technique of calculating geodesics. Conversely, extrensic methods, which use an external reference, such as a coordinate system, employs standard mathematical techniques to calculate geodesics.

Intrinsic methods typically treat the manifold as a vector field, which is a representation of the rates of change in all directions, and use the metric tensor and christoffel symbols, to calculate this rate of change w.r.t each direction, and how basis vectors change along the surface. This essay shall attempt to integrate the intrinsic and extrinsic methods to better understand the context in which they can be used. The Earth shall be modeled as a sphere within the orthogonal 3D Space, and the metric tensor will be used to calculate geodesics with respect to the origin in terms of the azimuthal and polar angles.

We use the notation $\mathbb{R}^3$ to represent our 3D space.

Definition 1.1 (Vector Space[4])**.** A *vector space* is a closed set with elements that can be added, scaled and satisfies the most basic vector properties, such as the commutative and associative properties of addition, and so on.

Definition 1.2 (Normed Vector Space[5])**.** A *normed vector space* takes the idea of a vector space and assigns a norm $\|\cdot\|$, that assigns a magnitude or size to each element, satisfying:

1. $\|v\| \geq 0$, and $\|v\| = 0$ if $v = 0$
2. $\|av\| = |a|\|v\|$ for all scalars a
3. $\|u + v\| \leq \|u\| + \|v\|$

[4]Definition adapted from: "Vector Space." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource, 10 Jan. 2009, `mathworld.wolfram.com/VectorSpace.html`.

[5]Ibid.

Using the Pythagorean theorem, in $\mathbb{R}^3$ the Euclidean norm is:

$$\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Definition 1.3 (Continuous Function Space[6])**.** The space $\mathcal{C}^0(x_1, x_2)$ consists of all functions $y : [x_1, x_2] \to \mathbb{R}$ that are continuous on $[x_1, x_2]$. That is,

$$y(x) \in \mathcal{C}^0(x_1, x_2) \quad \text{if} \quad \forall x \in [x_1, x_2], \quad \lim_{h \to 0} y(x + h) = y(x).$$

Definition 1.4 (N-Differentiable Function Space[7])**.** The space $\mathcal{C}^n(x_1, x_2)$ consists of all functions $y : [x_1, x_2] \to \mathbb{R}$ that are differentiable up to order n. That is,

$$y(x) \in \mathcal{C}^n(x_1, x_2) \quad \text{if} \quad y, y', y'', \ldots, y^{(n)} \text{ exist.}$$

For example, $\mathcal{C}^1(x_1, x_2)$ represents all functions that are differentiable up to the first order, within the interval $[x_1, x_2]$.

Remark. Clearly, $\mathcal{C}^n(x_1, x_2) \subset \mathcal{C}^{n-1}(x_1, x_2) \subset \cdots \subset \mathcal{C}^1(x_1, x_2) \subset \mathcal{C}^0(x_1, x_2)$, meaning that every function in $\mathcal{C}^n(x_1, x_2)$ is also in $\mathcal{C}^0(x_1, x_2)$, but not necessarily the other way around.

[6]Definition adapted from: MathTeacher. "Intro to Space of Continuous and Continuously Differentiable Functions." YouTube, March 4, 2022, `https://youtube.com/watch?v=o1SIHCq10EU`.

[7]Ibid.

2 Prerequisites

2.1 Multivariable Calculus

Multivariable calculus is an extension of single-variable calculus to functions of multiple variables, such as $f(x, y)$, $f(x, y, z)$, or more complex functions. It allows to understand how functions react to changes in a particular parameter, while other parameters are held constant. Graphically, it measures the rate of change of a function or a curve in the x, y, or z unit directions, known as a partial derivative. Taking the partial derivative of $f(x, y, z)$ with respect to x would involve differentiating it w.r.t. x, while treating other parameters as constants. Therefore, for a function $f(x, y, z)$, the partial derivative of f with respect to x will be[8]:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

Example. Let us consider the function $f(x, y, z) = x^2 + 3yz^2 + z^3$. The partial derivatives with respect to x, y, and z respectively will be:

$$\frac{\partial f}{\partial x} = 2x$$
$$\frac{\partial f}{\partial y} = 3z^2$$
$$\frac{\partial f}{\partial z} = 6yz + 3z^2$$

Remark. Notice that in the partial derivative with respect to x, y or z, the other two parameters are treated as constants.

2.1.1 Multivariable Chain Rule

The multivariable chain rule is an extension of the chain rule from single-variable calculus, allowing us to differentiate composites of functions where the input parameters

[8] "Partial Derivative." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource, 25 Nov. 2008, `mathworld.wolfram.com/PartialDerivative.html`.

themselves are functions of other parameters.

If $f(x, y, z)$ is a differentiable function where x, y, and z are themselves functions of a parameter t, i.e., $x = x(t)$, $y = y(t)$, and $z = z(t)$, the derivative of $f(x, y, z)$ with respect to t is given by[9]:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

More generally, if f is a function of n variables $x_1, x_2, \ldots, x_n$, each of which depends on t, then:

$$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt}.$$

2.1.2 Leibniz's Integral Rule

Leibniz's Integral Rule, or differentiation-under-the-integral-sign-rule, allows differentiation to pass inside an integral when the integrand depends on a parameter x.

Property 2.1 (Leibniz's Integral Rule[10])**.** *Let $f(x, t)$ be a function such that it is continuous and differentiable with respect to x for all t, and the integral*

$$F(x) = \int_{t_1}^{t_2} f(x, t)\, dt$$

is well-defined for all x. Liebniz's Integral Rule states the following:

$$\frac{d}{dx}\int_{t_1}^{t_2} f(x, t)\, dt = \int_{t_1}^{t_2} \frac{\partial f}{\partial x}\, dt$$

Remark. This formula simplifies the differentiation process in the 3D space by allowing us to move the derivative inside the integral when the integrand is parameterized.

[9] "14.5: The Chain Rule for Multivariable Functions." Mathematics LibreTexts, 5 Feb. 2025, https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/14\%3A_Differentiation_of_Functions_of_Several_Variables/14.05\%3A_The_Chain_Rule_for_Multivariable_Functions.

[10] Harron, Rob. "MAT-203 : The Leibniz Rule." math.hawaii.edu/rharron/teaching/MAT203/LeibnizRule.pdf.

2.2 Variational Calculus

The calculus of variations is a field of mathematical analysis that involves itself with functions that maximize or minimize functionals. It is to be noted that functionals map a function to a scalar, while functions map scalars onto scalars:

(Real) Functions:

$$f : \mathbb{R} \to \mathbb{R}$$

(Real) Functionals:

$$J : \mathcal{C}^0(a, b) \to \mathbb{R}$$

While differential calculus explores how changes in some parameter x can affect some function $y(x)$, variational calculus explores how changes in the path $y(x)$ between two points, affects the functional $J[y]$. For this essay, and in many applications, variational calculus is used to find this optimal 'path' between two points.

Definition 2.2 (Functional[11])**.** A functional is a correspondance that assigns a definite (real) number to a function or curve in some *normed linear space* $\mathcal{C}^0(a, b)$, that is $J[y(x)] : y(x) \to \mathbb{R}$, where $y(x)$ is an element of the space $\mathcal{C}^1$. The functional input is generally refered to as the path.

Definition 2.3 (Lagrangian[12])**.** Generally and for this essay, a functional can be formally defined as the expression:

$$J[y(x)] = \int_{x_a}^{x_b} F(x, y(x), y'(x))dx \tag{2.2.1}$$

where $y(x)$ varies over a space of differentiable functions on [a,b]. This integrand is called the Lagrangian, and it depends on the independent variable x, the function $y(x)$, and its derivative $y'(x)$, where $J \in \mathcal{C}^1[x_a, x_b]$.

Remark. Note that the definition uses only $y(x)$ as a parameter, while the Lagrangian accepts x, $y(x)$ and $y'(x)$. As x is implied, and the derivative can be determined

[11] Dass, H. K. Advanced Engineering Mathematics. S. Chand Publishing, 2007.
[12] Ibid.

exactly using the function, this simplification is valid.

When differentiating a function over a domain, a turning point is a point at which the graph of $y = f(x)$ is either at its minimum or maximum value, found by setting $f'(x) = 0$. Similarly, since functionals map functions onto a real numbers, they are also able to be extremized.

Definition 2.4 (Optimum path[13])**.** An optimum path is a solution to a given functional that yields either a minimum or a maximum. It is the function $y(x)$ that makes the functional $J[y(x)]$ stationary, and represents the extremum of the functional.

Definition 2.5 (Varied path[14])**.** A varied path $\overline{y}(x)$ (also called a variation of the path) is a function that is a small perturbation or modification of the optimal path. For a given value, the variation is the difference between the optimal path and the varied path.

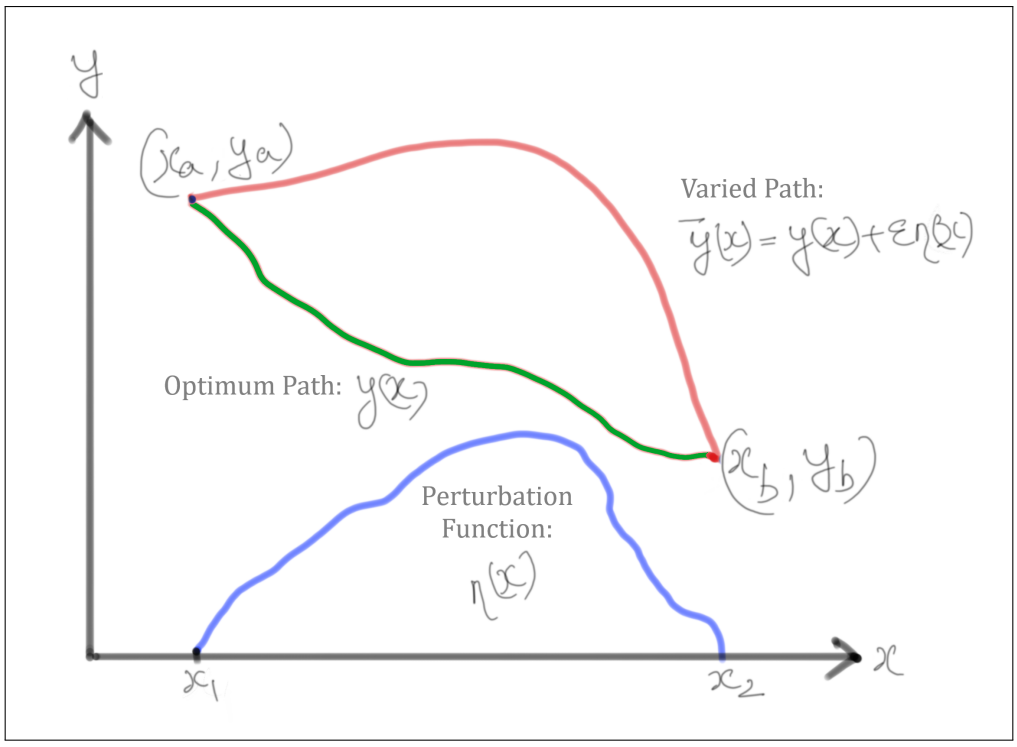

Figure 3: The varied path, $\bar{y}(x)$, can be thought of as the optimum path, $y(x)$, plus some small perturbation function, $\eta(x)$, s.t. the lower and upper bounds are constant.

[13]Good Vibrations with Freeball. "Introduction to Variational Calculu" YouTube, October 22, 2020, https://www.youtube.com/watch?v=6HeQc7CSkZs.

[14]Ibid.

Let the variation be defined by the following parameters:

* $y(x)$: the optimum path between the two points.
* $\eta(x)$: the variation function, chosen such that it vanishes at the endpoints. $\eta(x) \in \mathcal{C}^2$
* ε: a small variable multiplicative parameter, that controls the magnitude of the variation.

such that the following equation describes their relation.

$$\overline{y}(x) = y(x) + \varepsilon\eta(x) \tag{2.2.2}$$

Corollary 2.5.1. Differentiating both sides of the equation for the varied path, with respect to x:

$$\overline{y}'(x) = y'(x) + \varepsilon\eta'(x) \tag{2.2.3}$$

Condition 2.6 (Boundary Condition[15])**.** Since $y(x)$ and $\overline{y}(x)$ must pass through the same start (x_a, y_a) and end (x_b, y_b) point, the variation at the points are zero:

$$\eta(x_a) = \eta(x_b) = 0$$

Property 2.7. *For an optimal path $y(x)$[16]:*

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

Proof. It is expected that as $\varepsilon \to 0$, $\overline{y}(x) \to y(x)$, by definition 2.8. Since this is the optimum path, the Lagrangian must be at its stationary point, and thus its rate of change with respect to ε must equal 0.

[15]Ibid.
[16]Ibid.

2.3 Euler-Lagrange Equation

Attempting to solve the tautochrone problem, Leonhard Euler and Joseph-Louis Lagrange worked together to develop the Euler-Lagrange equation, a system of second-order differential equations whose solutions provide stationary points for a given functional, thus invaluable in optimization problems like the geodesic. The equation simplifies Property 2.9 by evaluating the functional as the Lagrangian (Definition 2.5) as follows:

$$\frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_{x_a}^{x_b} F(x, y(x), y'(x))dx\Big|_{\varepsilon=0} = 0$$

Applying Leibniz's Integral Rule, and converting $y(x)$, $y'(x)$ to $\overline{y}(x)$, $\overline{y}'(x)$, as they are equal at $\varepsilon = 0$:

$$\frac{d}{d\varepsilon}\int_{x_a}^{x_b} F(x, y(x), y'(x))dx\Big|_{\varepsilon=0} = \int_{x_a}^{x_b} \frac{\partial}{\partial\varepsilon}F(x, \overline{y}(x), \overline{y}'(x))\,dx\Big|_{\varepsilon=0} = 0$$

Using the multivariable chain rule:

$$\int_{x_a}^{x_b} \frac{\partial}{\partial\varepsilon}F(x, \overline{y}(x), \overline{y}'(x))\,dx\Big|_{\varepsilon=0} = \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial x}\frac{dx}{d\varepsilon} + \frac{\partial F}{\partial \overline{y}}\frac{d\overline{y}}{d\varepsilon} + \frac{\partial F}{\partial \overline{y}'}\frac{d\overline{y}'}{d\varepsilon}\right)dx\Big|_{\varepsilon=0} = 0$$

$$\implies \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial x}\cancelto{0}{\frac{dx}{d\varepsilon}} + \frac{\partial F}{\partial \overline{y}}\frac{d\overline{y}}{d\varepsilon} + \frac{\partial F}{\partial \overline{y}'}\frac{d\overline{y}'}{d\varepsilon}\right)dx\Big|_{\varepsilon=0} = 0 \quad \text{(As } x \text{ is independent of } \varepsilon\text{)}$$

Corollary 2.7.1 (Partial derivatives of varied path)**.** From Definition 2.7 and Corollary 2.7.1:

$$\overline{y}(x) = y(x) + \varepsilon\eta(x) \implies \frac{\partial\overline{y}}{\partial\varepsilon} = \eta(x)$$
$$\overline{y}'(x) = y'(x) + \varepsilon\eta'(x) \implies \frac{\partial\overline{y}'}{\partial\varepsilon} = \eta'(x)$$

Plugging in Corollary 2.9.1:

$$\implies \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial \overline{y}}\eta + \frac{\partial F}{\partial \overline{y}'}\eta'\right)dx\Big|_{\varepsilon=0} = 0$$

At $\varepsilon = 0$, $\overline{y}(x) = y(x)$ and $\overline{y}'(x) = y'(x)$

$$\implies \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right) dx = 0$$
$$\implies \int_{x_a}^{x_b} \frac{\partial F}{\partial y}\eta \, dx + \int_{x_a}^{x_b} \frac{\partial F}{\partial y'}\eta' \, dx = 0$$

The equation is now ready to be integrated by parts, setting $\frac{\partial F}{\partial y'}$ as u and η' as dv:

$$\int_{x_a}^{x_b} \frac{\partial F}{\partial y}\eta \, dx + \int_{x_a}^{x_b} \frac{\partial F}{\partial y'}\eta' \, dx = 0$$
$$\implies \int_{x_a}^{x_b} \frac{\partial F}{\partial y}\eta \, dx + \cancelto{0}{\left[\frac{\partial F}{\partial y'}\eta(x) \right]_{x_a}^{x_b}} - \int_{x_a}^{x_b} \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\eta \, dx = 0 \quad \text{(Condition 2.8)}$$
$$\implies \int_{x_a}^{x_b} \frac{\partial F}{\partial y}\eta \, dx - \int_{x_a}^{x_b} \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\eta \, dx = 0$$
$$\implies \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y}\eta - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\eta \right) dx = 0$$
$$\implies \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right)\eta \, dx = 0$$

For the above equation to hold for all functions $\eta(x)$, the term in the brackets must be zero independent of the arbitrary function:

Property 2.8 (The Euler-Lagrange equation[17])**.** *For a Jacobian functional $J[y(x)]$ of the form:*

$$\int_a^b F(x, y(x), y'(x))dx$$

defined within the function space $y(x) \in \mathcal{C}^1[a, b]$ to have an extremum for a given function $y(x)$, it is necessary that $y(x)$ satisfy the Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0$$

[17]Dass, H. K. Advanced Engineering Mathematics. S. Chand Publishing, 2007.

3 Geodesics

3.1 Arc Length Integral

Let us suppose $\vec{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ is a smooth parametric curve, meaning it is differentiable and continuous in its domain, lying in $\mathbb{R}^3$, where t is varied over some interval $[a, b]$. In order to calculate the length L of this arc, the curve can be approximated by dividing the interval into n line segment subintervals as follows, where each t_i represents a point on the curve:

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b \tag{3.1.1}$$

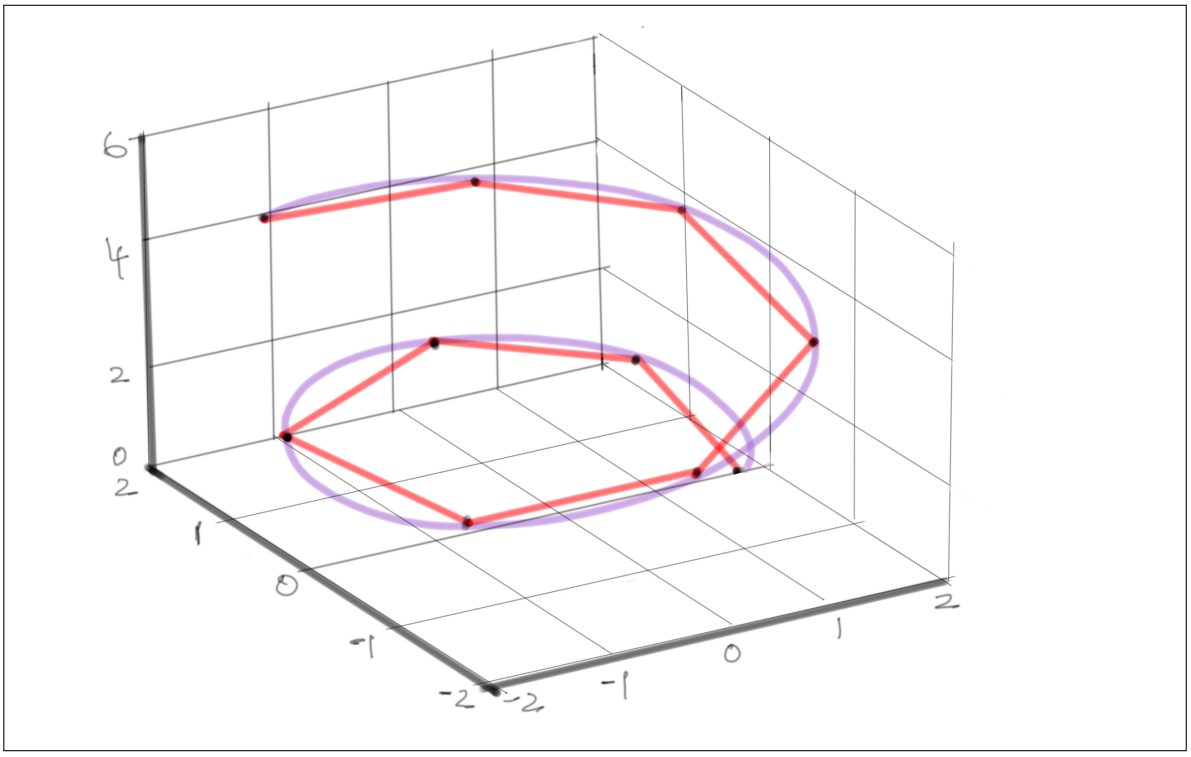

Figure 4: We divide the arc into n evenly distributed points connected by line segments, to approximate the length of the curve. As $n \to \infty$, the accuracy increases.

Thus, the distance of the line segment between two successive points t_{i-1} and t_i is given by:

$$\Delta L_i \approx \|\vec{r}(t_i) - \vec{r}(t_{i-1})\|$$

$$
\begin{aligned}
&= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} \quad \text{(Using Pythagoras' theorem in 3D)} \\
&= \frac{\Delta t}{\Delta t}\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} \quad \text{(Multiplying by } \frac{\Delta t}{\Delta t} = 1\text{)} \\
&= \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 + \left(\frac{\Delta z_i}{\Delta t}\right)^2} \cdot \Delta t
\end{aligned}
$$

We can estimate length of the curve by adding the n intervals to create a segmented approximation:

$$
\begin{aligned}
L &\approx \sum_{i=1}^{n} \Delta L_i \\
&= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 + \left(\frac{\Delta z_i}{\Delta t}\right)^2} \cdot \Delta t
\end{aligned}
\tag{3.1.2}
$$

The expression can now be viewed as a Riemann sum, where the quantity inside the square root represents the rate of change in each spatial direction over the parameter t, and Δt represents the width of each subinterval. As $n \to \infty$, $\Delta t \to 0$, and the sum becomes an exact representation of the arc length:

$$
L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 + \left(\frac{\Delta z_i}{\Delta t}\right)^2} \cdot \Delta t
$$

The Riemann Sum converges to the definite integral, and the discrete differences Δx_i, Δy_i, and Δz_i become infinitesimal, defined on the bounds from $t = a$ to $t = b$

Property 3.1 (Arc Length of a Smooth Parametric Curve[18])**.** *Let $r(t) = (x(t), y(t), z(t))$ be a continuously differentiable function in $\mathbb{R}^3$ on the interval $t \in [a, b]$. Then, the arc length of $r(t)$ over this interval is given by:*

$$
L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \tag{3.1.3}
$$

Remark. We notice several important properties. First, the integral depends on the derivatives of $x(t)$, $y(t)$, and $z(t)$ with respect to some definitive parameter t, which

[18]Dr. Trefor Bazett. "How long is a curve?? The Arclength Formula in 3D." YouTube, October 14, 2019, `https://youtube.com/watch?v=80J5s0pic8M`.

means the curve must be smooth for the integral to be well-defined. Most importantly, this integral represents a functional, as it takes the function $\vec{r}(t)$ as input and maps it to a real scalar value, and can be minimized using the Euler-Lagrange equation, to calculate minimum length curves on the manifold in $\mathbb{R}^3$.

The Arc Length Integral is the reason calculating geodesics extrinsically results in easier mathematical calculations than intrinsic calculations.

3.2 Deriving the Geodesic Equations

Definition 3.2 (Geodesic[19])**.** A **geodesic** on a surface is the shortest path between two nearby points, generalizing the notion of a straight line to curved spaces. Mathematically, it is a curve whose tangent vector remains parallel to itself, meaning it experiences no local (tangential) acceleration.

The vector equation represented by $\vec{r}(t) = r_x(u,v)\hat{\mathbf{i}} + r_y(u,v)\hat{\mathbf{j}} + r_z(u,v)\hat{\mathbf{k}}$ essentially conveys that the u-v parametric plane is being transformed onto a curved surface like a sphere, arbtirarily mapping a cartesian plane onto a curvilinear manifold.

A line in our parametric space can be represented by $u = u(t)$ and $v = v(t)$, where t is our single varied parameter.

Substituting this equation for a line into our equation for the surface:

$$\vec{r}(t) = x(u(t), v(t))\hat{\mathbf{i}} + y(u(t), v(t))\hat{\mathbf{j}} + z(u(t), v(t))\hat{\mathbf{k}}$$

Thus, a geodesic curve from a to b would minimize the distance of this curve, given by the arc length integral:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Applying the Multivariable Chain Rule:

[19] Weinstock, Robert. "Calculus of Variations - With Applications to Physics and Engineering (1974)", Read Books, 2008.

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt} \\
\implies \left(\frac{dx}{dt}\right)^2 &= \left(\frac{\partial x}{\partial u}\right)^2\left(\frac{du}{dt}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\left(\frac{dv}{dt}\right)^2 + 2\frac{\partial x}{\partial u}\frac{du}{dt}\frac{\partial x}{\partial v}\frac{dv}{dt}\frac{dx}{dt} \\
\frac{dy}{dt} &= \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} \\
\implies \left(\frac{dy}{dt}\right)^2 &= \left(\frac{\partial y}{\partial u}\right)^2\left(\frac{du}{dt}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2\left(\frac{dv}{dt}\right)^2 + 2\frac{\partial y}{\partial u}\frac{du}{dt}\frac{\partial y}{\partial v}\frac{dv}{dt}\frac{dy}{dt} \\
\frac{dz}{dt} &= \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} \\
\implies \left(\frac{dz}{dt}\right)^2 &= \left(\frac{\partial z}{\partial u}\right)^2\left(\frac{du}{dt}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\left(\frac{dv}{dt}\right)^2 + 2\frac{\partial z}{\partial u}\frac{du}{dt}\frac{\partial z}{\partial v}\frac{dv}{dt}\frac{dz}{dt}
\end{aligned} \tag{3.2.1}$$

Inserting equation (3.2.1) into equation (3.1.3) and simplifying by grouping like terms:

$$\begin{aligned}
L = \int_a^b \Bigg\{ &\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right]\left(\frac{du}{dt}\right)^2 \\
&+ 2\left[\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right]\frac{du}{dt}\frac{dv}{dt} \\
&+ \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]\left(\frac{dv}{dt}\right)^2\Bigg\}^{\frac{1}{2}} dt.
\end{aligned} \tag{3.2.2}$$

While simply using these formulas as a basis for the Euler-Lagrange equation is sufficient, not only does this cause extremely lengthy calculations, but it also causes a heavy deal of abstraction. To alleviate this, let us ground ourself by developing an understanding of the Non-Euclidean nature of our curvilinear surface itself.

3.3 Distances on a Surface

Generally, for 3D vectors in our maths lessons, we have applied the Pythagorean theorem to calculate the magnitude using the length of its components:

$$\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

However, this relies on the fact that the vectors along coordinate axes are perpendicular to each other, which is not the case in our non-Euclidean geometry, as the vectors of the reference frame can form any angle along the surface. In this essay, these vectors will be termed as reference vectors.

Instead let us attempt to use the vector dot product formula to calculate the length. Considering an arbitrary vector $\vec{ds}$ in a coordinate system where the manifold is represented as a 2D surface, with reference vectors $\vec{r_u}$ and $\vec{r_v}$, since this manifold is made up of twists and turns, these vectors are not orthogonal, and can change during movement along the surface. Taking one point on this coordinate system, since the reference vectors can be computed at this point, distances can be represented as a relationship between these reference vectors. Considering an infinitesimal length ds on the plane, the below triangle is formed:

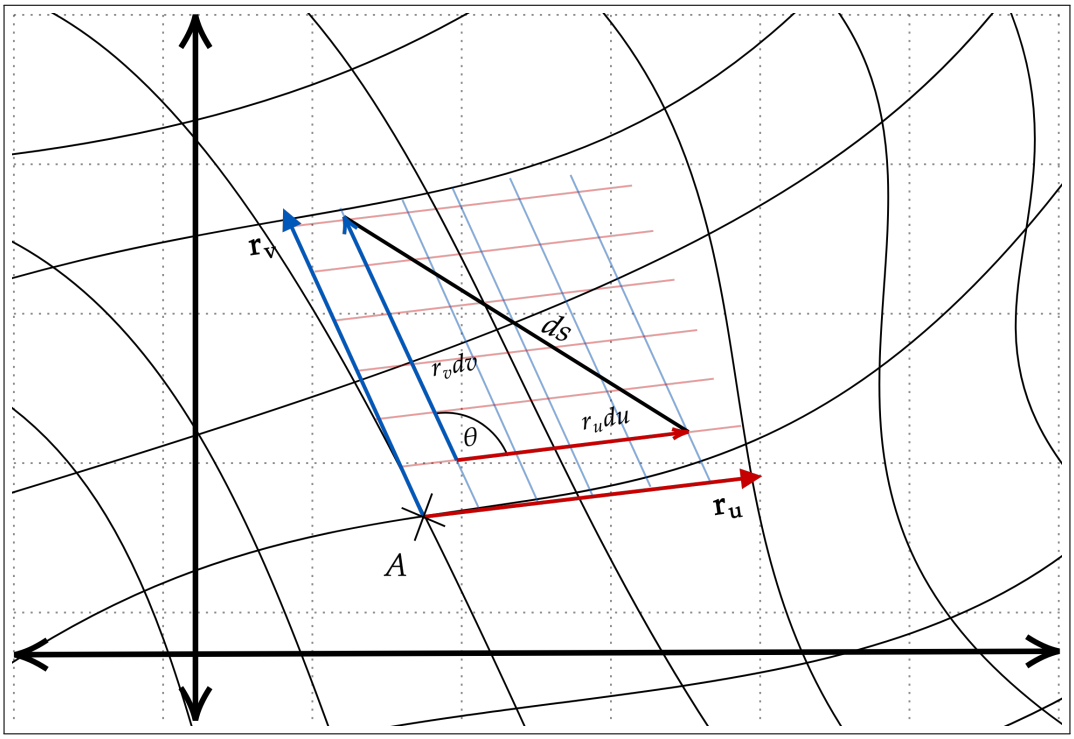

Figure 5: The curvilinear system at point A has tangent vectors r_v and r_u, which forms a skewed coordinate system with its own calculations for distances within itself.

By using the cosine rule:

$$\begin{aligned} ds^2 &= |\vec{r}_u\,du|^2 + |\vec{r}_v\,dv|^2 - 2\,|\vec{r}_u\,du|\,|\vec{r}_v\,dv|\cos\theta \\ &= |\vec{r}_u\,du|^2 + |\vec{r}_v\,dv|^2 - 2\,|\vec{r}_u\,du|\,|\vec{r}_v\,dv|\cos\left(180^\circ - \theta\right) \\ &= |\vec{r}_u\,du|^2 + |\vec{r}_v\,dv|^2 + 2\,|\vec{r}_u\,du|\,|\vec{r}_v\,dv|\cos\theta \\ &= (\vec{r}_u\,du)\cdot(\vec{r}_u\,du) + (\vec{r}_v\,dv)\cdot(\vec{r}_v\,dv) + 2\,(\vec{r}_u\,du)\cdot(\vec{r}_v\,dv) \\ &= (\vec{r}_u\cdot\vec{r}_u)\,du^2 + (\vec{r}_v\cdot\vec{r}_v)\,dv^2 + 2(\vec{r}_u\cdot\vec{r}_v)\,du\,dv \end{aligned} \tag{3.3.1}$$

Thus, if we have the dot product of these vectors, which simply means the tangent vectors along the manifold at every point along the surface, we have the ability to translate distances from our parametric space to the manifold. This representation is called the metric tensor, and it allows us to translate coordinate distances, which exist in our superficial frames, to proper distances, which are globalized.

Definition 3.3 (Tensor[20])**.** A tensor is a mathematical object that can be thought of as a generalization of scalars, vectors, and matrices. It's akin to a container that stores data in a structured way. They are used to represent things like physical quantities (force, velocity, stress etc.).

In the 19th century, Carl Friedrich Gauss studied curved surfaces and wanted to measure distances, angles, and areas in a way that remained unchanged if the surface was bent but not stretched[21]. He described surfaces using two parameters, u and v, and realized that these measurements depended on how the surface was stretched or compressed.

To formalize this, he introduced the metric tensor, which translates distances from our parametric coordinate frame to the manifold. It can be represented through various notations:

[20]Definition adapted from: Veblen, Oswald, and J. H. Whitehead. The Foundations of Differential Geometry. 1953.

[21]Gauss, Carl Friedrich. "General Investigations of Curved Surface". Translated by A. M. Hiltebeitel and J. C. Morehead, Raven Press, 1965.

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} r_{uu} & r_{uv} \\ r_{vu} & r_{vv} \end{bmatrix} = \begin{bmatrix} \vec{r_u} \cdot \vec{r_u} & \vec{r_u} \cdot \vec{r_v} \\ \vec{r_v} \cdot \vec{r_u} & \vec{r_v} \cdot \vec{r_v} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Since the dot product satisfies the commutative property, the metric tensor is symmetric, in that r_{uv} equals r_{vu}. Reverting back to the fundamental ideas of the metric tensor, we know that it is simply the tangent vector at each point of the curve, with respect to the parameters. Thus, we can represent these reference vectors using the partial derivatives:

$$\vec{r_u} = \frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$$

$$\vec{r_v} = \frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

Then, the metric tensor components are as follows:

$$\begin{aligned} E &= \vec{r_u} \cdot \vec{r_u} = \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2 \\ F &= \vec{r_u} \cdot \vec{r_v} = \left(\frac{\partial x}{\partial u} \right) \left(\frac{\partial x}{\partial v} \right) + \left(\frac{\partial y}{\partial u} \right) \left(\frac{\partial y}{\partial v} \right) + \left(\frac{\partial z}{\partial u} \right) \left(\frac{\partial z}{\partial v} \right) \\ G &= \vec{r_v} \cdot \vec{r_v} = \left(\frac{\partial x}{\partial v} \right)^2 + \left(\frac{\partial y}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2 \end{aligned} \tag{3.3.2}$$

Plugging these values into equation (3.2.2), for the arc length integral, a simplifcation can be carried out for a less daunting expression with components of the metric tensors, which is a functional that can be minimized by the Euler-Lagrange equation:

$$L = \int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2} dt \tag{3.3.3}$$

Remark. For cleaner notation, the substitution $u' = \frac{du}{dt}$ and $v' = \frac{dv}{dt}$ has been performed.

3.4 Geodesic Equations for a Sphere

On Earth, we describe positions on the surface using the latitudinal and longitudinal coordinates. Generally, we call these the **azimuthal** and **polar** angle, where the azimuthal is the angle measured counterclockwise from the positive x-axis in the xy-plane, and the polar angle is the angle measured from the positive z-axis downward toward the point.

The Earth is an oblate spheroid, not a perfect sphere, as centrifugal force pushes the Earth's mass outwards, causing the equitorial diameter to be larger than the polar diameter, creating an ellipsoid. Using data from the WGS84[22] (World Geodetic System):

$$\text{Semi Major Axis } (a) = 6,378,137.0\ m$$

$$\text{Semi Minor Axis } (b) \approx 6,356,752.314245\ m$$

Just by inspection, we can see the difference between the axes, relative to the scale of the Earth, seems to be negligible. We can show this by measuring the "flattening" of the ellipsoid:

$$f = \frac{a-b}{a} \approx \frac{1}{298.2572235604} \approx 0.0033528106$$

Thus, for this essay, we shall model the Earth as a sphere, but it must be noted that for official GPS and Navigational calculations, the ellipsoidal model of the Earth is used to maintain accuracy.

Let us place a sphere in our $\mathbb{R}^3$ space, with dimensions of the earth such that:

$$Radius of the Earth = 6,378,137.0\ m$$

$$\theta = \text{Azimuthal angle} = \text{Angle of Longitude}$$

$$\phi = \text{Polar angle} = \text{Angle of Latitude}$$

[22] "World Geodetic System 1984 (WGS 84)". Office of Geomatics, National Geospatial-Intelligence Agency.

The surface of a sphere is given by the following parametric equation, which is intuitively similar to our understanding of the unit circle in trigonometry:

$$x = R\sin\phi\cos\theta$$

$$y = R\sin\phi\sin\theta$$

$$z = R\cos\phi$$

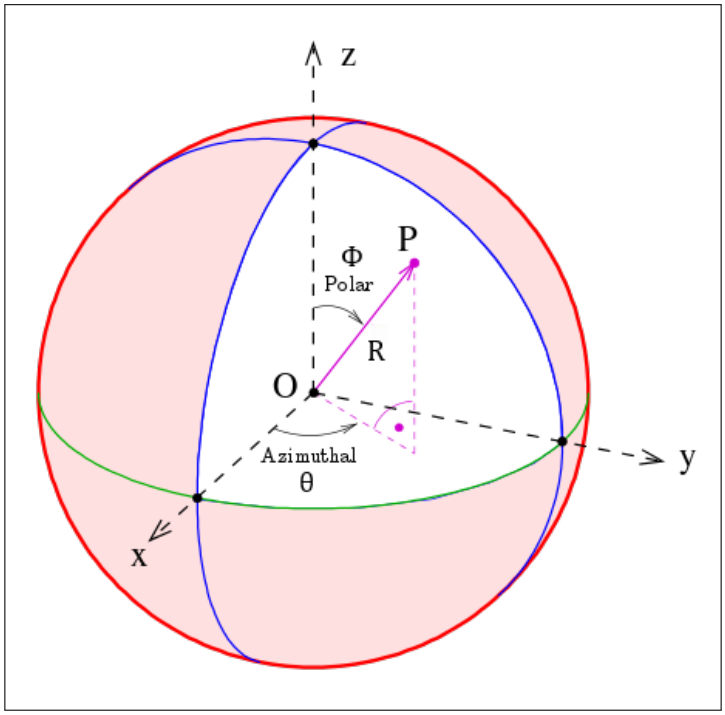

Figure 6: Spherical coordinates, rather than cartesian coordinates, are used to represent points on a sphere, with variables as represented above. [vvvv.org]

Now, we can calculate the components of the Metric Tensor E, F, and G, also known as the First Fundamental Form, of our curve, using equation (3.3.2) and take the partial derivatives.

$$\begin{aligned} G &= \left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 \\ &= (-R\sin\phi\sin\theta)^2 + (R\sin\phi\cos\theta)^2 + 0 \\ &= R^2\left(\sin^2\theta\sin^2\phi + \sin^2\phi\cos^2\theta\right) \\ &= R^2\sin^2\phi\left(\sin^2\theta + \cos^2\theta\right) \\ &= R^2\sin^2\phi \end{aligned} \tag{3.4.1}$$

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 \\ &= \left(R\cos\theta\cos\phi\right)^2 + \left(R\sin\theta\cos\phi\right)^2 + \left(-R\sin\phi\right)^2 \\ &= R^2\left(\cos^2\theta\cos^2\phi + \sin^2\theta\cos^2\phi + \sin^2\phi\right) \\ &= R^2\left(\cos^2\phi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\phi\right) \\ &= R^2\left(\cos^2\phi + \sin^2\phi\right) \\ &= R^2 \end{aligned} \tag{3.4.2}$$

$$\begin{aligned} F &= \left(\frac{\partial x}{\partial \phi}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial \phi}\right)\left(\frac{\partial y}{\partial \theta}\right) + \left(\frac{\partial z}{\partial \phi}\right)\left(\frac{\partial z}{\partial \theta}\right) \\ F &= -\left(R\cos\theta\cos\phi\right)\left(R\sin\phi\sin\theta\right) \\ &\quad + \left(R\sin\theta\cos\phi\right)\left(R\sin\phi\cos\theta\right) + 0 \\ &= 0 \end{aligned} \tag{3.4.3}$$

Using these values, we not only compute the Metric Tensor for the sphere, given by the following, but we can further plug these values into our Arc Length Formula.

$$\text{Metric Tensor of a Sphere} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\phi \end{bmatrix} \tag{3.4.4}$$

While writing it in this form is not necessary for the following calculations, and the first fundamental form is sufficient, it helps us understand the simple nature of the intrinsic geometry of the surface, which in other applications of Geodesics becomes the fundamental tool to understand any manifold.

The distance along this sphere from t_a to t_b, the parametric boundaries, is:

$$L = \int_{t_a}^{t_b} \sqrt{R^2 \times \phi'^2 + 2 \times 0 \times \phi'\theta' + R^2\sin^2\phi \times \theta'^2}\, d\phi \tag{3.4.5}$$

Note that when we are creating a path on the surface, instead of parametrizing our equation in the form $\phi(t)$ and $\theta(t)$, we can treat one as a function of the other, akin to plotting a graph of $y = f(x)$ in a coordinate axis, or in this case, $\theta = \theta(\phi)$. This

can be shown with a simple change of variables, where we let $t = \phi$:

$$L = \int_{\phi_a}^{\phi_b} \sqrt{R^2 \left(\frac{\cancel{d\phi}}{\cancel{d\phi}}\right)^2 + R^2 \sin^2\phi \times \left(\frac{d\theta}{d\phi}\right)^2}\, d\phi$$

Factoring out R^2, and taking it out of the integral:

$$= R \int_{\phi_a}^{\phi_b} \sqrt{1 + \sin^2\phi \times \left(\frac{d\theta}{d\phi}\right)^2}\, d\phi \tag{3.4.6}$$

This is essentially in the form of the following Lagrangian:

$$= R \int_{\phi_a}^{\phi_b} F(\phi, \theta, \theta')\, d\phi$$

Remark. Since notation has been used quite liberally, acknowledge that θ' in (3.4.5) refers to $\frac{d\theta}{dt}$, while in (3.4.6) and onwards, it shall refer to $\frac{d\theta}{d\phi}$.

Let us apply the Euler-Lagrange equation, with respect to the two parameters $\theta = \theta(\phi)$ and ϕ, as we want to minimize the distance between the two points on our sphere. This results in only one Euler-Lagrange equation, as we may reduce the number of parameters in our curve to just ϕ:

$$\frac{\partial F}{\partial \theta} - \frac{d}{d\phi}\left(\frac{\partial F}{\partial \theta'}\right) = 0 \tag{3.4.7}$$

Firstly, note that in our integrand, there is no appearance of θ, and thus $\frac{\partial F}{\partial \theta}$ must equal 0, as everything else is treated as a constant. Consequently:

$$\begin{aligned} -\frac{d}{d\phi}\left(\frac{\partial F}{\partial \theta'}\right) &= 0 \\ \implies \frac{d}{d\phi}\left(\frac{\partial F}{\partial \theta'}\right) &= 0 \end{aligned} \tag{3.4.8}$$

Multiplying $d\phi$ on both sides, and integrating:

$$\begin{aligned} d\left(\frac{\partial F}{\partial \theta'}\right) &= 0 \\ \implies \frac{\partial F}{\partial \theta'} &= c_1 \end{aligned} \tag{3.4.9}$$

Evaluating the expression for $\frac{\partial F}{\partial \theta'}$:

$$\begin{aligned}\frac{\partial F}{\partial \theta'} &= \frac{\partial}{\partial \theta'}\left(\sqrt{1+\theta'^2\sin^2\phi}\right) \\ &= \frac{\partial}{\partial \theta'}\left(\sqrt{1+\theta'^2\sin^2\phi}\right)\end{aligned} \tag{3.4.10}$$

Using the power rule and the chain rule, we can further simplify this, treating $\sin^2\phi$ as a constant

$$\begin{aligned}\frac{\partial F}{\partial \theta'} &= \frac{\partial}{\partial \theta'}\left(1+\theta'^2\sin^2\phi\right)^{\frac{1}{2}} \\ &= \frac{1}{\cancel{2}} \times \left(1+\theta'^2\sin^2\phi\right)^{-\frac{1}{2}} \times \cancel{2} \times \sin^2\phi \times \theta' \\ &= \frac{\theta'\sin^2\phi}{\sqrt{1+\theta'^2\sin^2\phi}} = c_1\end{aligned} \tag{3.4.11}$$

Now, by solving this differential equation, we can express θ in terms of ϕ, and finally obtain the equation for the geodesic of the sphere. Performing some algebra to express θ' in terms of ϕ and c_1:

$$\begin{aligned}&\frac{\theta'\sin^2\phi}{\sqrt{1+\theta'^2\sin^2\phi}} = c_1 \\ \implies &\frac{\theta'^2\sin^4\phi}{1+\theta'^2\sin^2\phi} = {c_1}^2 \quad \text{[Squaring both sides]} \\ \implies &\theta'^2\sin^4\phi = {c_1}^2\left(1+\theta'^2\sin^2\phi\right) \\ \implies &\theta'^2\sin^4\phi = {c_1}^2 + {c_1}^2\theta'^2\sin^2\phi \quad \text{[Distributing]} \\ \implies &\theta'^2\sin^4\phi - {c_1}^2\theta'^2\sin^2\phi = {c_1}^2 \\ \implies &\theta'^2\left(\sin^4\phi - {c_1}^2\sin^2\phi\right) = {c_1}^2 \quad \text{[Combining like terms]} \\ \implies &\theta'^2 = \frac{{c_1}^2}{\sin^4\phi - {c_1}^2\sin^2\phi} \\ \implies &\theta'^2 = \frac{{c_1}^2}{\sin^4\phi - {c_1}^2\sin^2\phi}\end{aligned}$$

While these differential equations will have to be solved (to obtain c_1), the obtained expression for θ'^2 (3.4.12) can be plugged back into our Lagrangian (3.4.6), and

integrated under the bounds of ϕ_a and ϕ_b. Thus:

$$L = R\int_{\phi_a}^{\phi_b} \sqrt{1+\sin^2\phi \times \left(\frac{{c_1}^2}{\sin^4\phi - {c_1}^2\sin^2\phi}\right)}d\phi$$

Cancelling out the powers of $\sin\phi$:

$$= R\int_{\phi_a}^{\phi_b} \sqrt{1+\frac{{c_1}^2}{\sin^2\phi - {c_1}^2}}d\phi$$

Bringing fractions to a common denominator:

$$= R\int_{\phi_a}^{\phi_b} \sqrt{\frac{\sin^2\phi - \cancel{c_1^2} + \cancel{c_1^2}}{\sin^2\phi - {c_1}^2}}d\phi$$

Factoring $\sin^2\phi$, and reducing the square root:

$$= R\int_{\phi_a}^{\phi_b} \frac{\sin\phi}{\sqrt{\sin^2\phi - {c_1}^2}}d\phi$$

Using the substitution $u = \cos\phi,$ so that $du = -\sin\phi\, d\phi$:
$\sin^2\phi = 1-u^2$, and bounds become $\cos(\phi_a) to \cos(\phi_b)$

$$= -R\int_{cos(\phi_a)}^{cos(\phi_b)} \frac{du}{\sqrt{(1-u^2)-c_1^2}}.$$

Simplifying the denominator:

$$= -R\int_{cos(\phi_a)}^{cos(\phi_b)} \frac{du}{\sqrt{1-c_1^2-u^2}}.$$

Recall that $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

Assuming $1-{c_1}^2 \geq 0$, we may take ${c_2}^2 = 1-{c_1}^2$:

Substituting ${c_2}^2 = 1-c_1^2$:

$$= -R\int_{\cos(\phi_a)}^{\cos(\phi_b)} \frac{du}{\sqrt{{c_2}^2-u^2}}.$$

$$= -R\arcsin\left(\frac{u}{c_2}\right)\Bigg|_{\cos(\phi_a)}^{\cos(\phi_b)}$$

$$= -R\left[\arcsin\left(\frac{\cos(\phi_b)}{c_2}\right) - \arcsin\left(\frac{\cos(\phi_a)}{c_2}\right)\right]$$

Remark. Note that since we are calculating a distance, ϕ_a and ϕ_b must be interchangeable and $L > 0$. Thus, we must take the absolute value of the expression.

Thus we have inferred that **distances on a sphere's geodesic** are given by:

Let L be the distance along a geodesic on a sphere of radius R, between (ϕ_a, θ_a) and (ϕ_b, θ_b), and c_1 a constant determined by the boundary conditions. Then,

$$L = R\left|\arcsin\left(\frac{\cos(\phi_a)}{c_2}\right) - \arcsin\left(\frac{\cos(\phi_b)}{c_2}\right)\right|, \tag{3.4.12}$$

where $c_2 = \sqrt{1 - c_1^2}$ and $c_1^2 \leq 1 \implies c_2^2 \leq 1$ for the solution to be valid.

3.5 Determining the Constant c_2

Our main issue lies in the variable c_1, which arose in equation (3.4.11) upon integration. As mentioned earlier, to evaluate this we must solve the differential equation to get the equation of our geodesic in $\mathbb{R}^3$.

$$\left(\frac{d\theta}{d\phi}\right)^2 = \frac{{c_1}^2}{\sin^4\phi - {c_1}^2\sin^2\phi}$$
$$\implies \left(\frac{d\theta}{d\phi}\right)^2 = \frac{c_1^2}{\sin^2\phi\left(\sin^2\phi - c_1^2\right)}$$
$$\implies \frac{d\theta}{d\phi} = \frac{c_1}{\sin\phi\sqrt{\sin^2\phi - c_1^2}}$$

Taking the positive square root:

$$\implies d\theta = \frac{c_1 d\phi}{\sin\phi\sqrt{\sin^2\phi - c_1^2}}$$

Integrating both sides:

$$\implies \theta = \int \frac{c_1 d\phi}{\sin\phi\sqrt{\sin^2\phi - c_1^2}}$$

Performing the substitution $u = c_1\cot\phi$ (so that $du = -c_1\csc^2\phi\, d\phi$), by constructing a right triangle with base angle ϕ, we find that, since $\cot\phi = \frac{u}{c_1} = \frac{adj}{opp}$, $\sin\phi = \frac{opp}{hyp} = \frac{c_1}{\sqrt{{c_1}^2+u^2}}$ and $\sin^2\phi = \frac{{c_1}^2}{{c_1}^2+u^2}$. Thus substituting:

$$\theta = \int \frac{\cancel{c_1}}{\cancel{\sin\phi}\sqrt{\frac{{c_1}^2}{{c_1}^2+u^2} - {c_1}^2}}\left(\frac{-\cancel{\sin^2\phi}}{\cancel{c_1}}\right) du$$

Factoring out c_1, and substituting further:

$$= -\int \frac{1}{\cancel{c_1}\sqrt{\frac{1}{{c_1}^2+u^2}-1}}\left(\frac{\cancel{c_1}}{\sqrt{{c_1}^2+u^2}}\right)du$$
$$= -\int \frac{1}{\sqrt{\frac{1}{{c_1}^2+u^2}-1}}\left(\frac{du}{\sqrt{{c_1}^2+u^2}}\right)$$

Simplifying further:

$$= -\int \frac{1}{\sqrt{\frac{1-{c_1}^2-u^2}{{c_1}^2+u^2}}}\left(\frac{du}{\sqrt{{c_1}^2+u^2}}\right)$$
$$= -\int \frac{\cancel{\sqrt{{c_1}^2+u^2}}}{\sqrt{1-{c_1}^2-u^2}}\left(\frac{du}{\cancel{\sqrt{{c_1}^2+u^2}}}\right)$$
$$= -\int \frac{du}{\sqrt{1-{c_1}^2-u^2}}$$

Performing the same substitution as earlier, i.e. ${c_2}^2 = 1-{c_1}^2$, and performing the integration using the property of $\frac{d}{dx}\arcsin\left(\frac{x}{a}\right)$:

$$\theta = -\arcsin\left(\frac{u}{c_2}\right)+c_3$$

Recalling that $\arcsin(x)+\arccos(x)=\frac{\pi}{2}$:

$$\implies \theta = \arccos\left(\frac{u}{c_2}\right)-\frac{\pi}{2}+c_3$$

Substituting $u = c_1\cot\phi$, and letting $c_3-\frac{\pi}{2}=\theta_0$:

$$\theta = \arccos\left(\frac{c_1\cot\phi}{c_2}\right)+\theta_0$$

Property 3.4 (Boundary Conditions[23])**.** *Given two points (ϕ_a,θ_a) and (ϕ_b,θ_b) lying on a geodesic, where ϕ denotes the polar angle and θ denotes the azimuthal angle, their coordinates satisfy the following boundary conditions:*

[23]Property adapted from: Faculty of Khan. "The Geodesic Problem on a Sphere — Calculus of Variations." YouTube, December 3, 2018, `https://youtube.com/watch?v=1cZC3iMpDxQ`.

$$\theta = \arccos\left(\frac{\sqrt{1-{c_2}^2}}{c_2}\cot\phi\right) + \theta_0.$$

The constants c_2, and θ_0 are determined by the conditions:

$$\theta_a = \arccos\left(\frac{\sqrt{1-{c_2}^2}}{c_2}\cot\phi_a\right) + \theta_0, \quad \theta_b = \arccos\left(\frac{\sqrt{1-{c_2}^2}}{c_2}\cot\phi_b\right) + \theta_0.$$

Upon subtracting the first equation from the second, the following equation is obtained, which is independent of θ_0:

$$\theta_b - \theta_a = \arccos\left(\frac{\sqrt{1-{c_2}^2}}{c_2}\cot\phi_b\right) - \arccos\left(\frac{\sqrt{1-{c_2}^2}}{c_2}\cot\phi_a\right)$$

This condition ensures that the geodesic passes through both specified points, satisfying the constraints imposed by the geodesic equations.

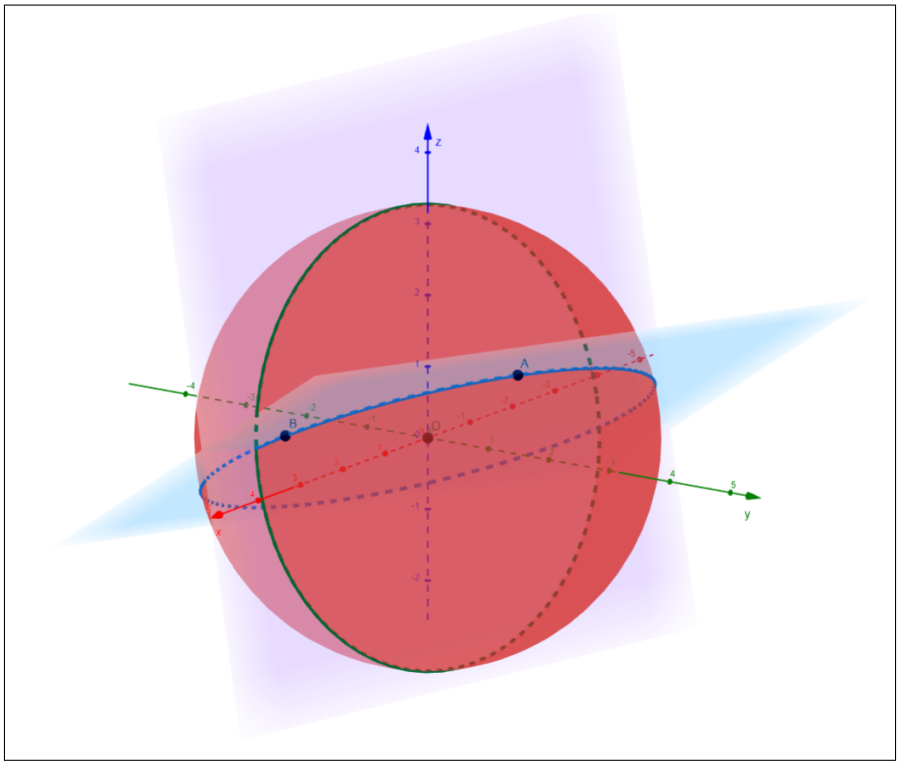

Figure 7: An intuitive understanding of the above geodesic equation is that it forms the intersection between a plane passing through the origin and the sphere, creating Great Circle arcs between points (equitorial radius). [GeoGebra]

By solving the above equation, we are able to derive the value of c_2, allowing us to

plug it into our equation for the distance (equation 3.4.12), and calculate the shortest distance between two points on a sphere. However, the very existence of a bound to c_2, becomes a limitation of the equation, as certain values of ϕ or θ do not result in a final answer. These limitations exist as the domain of the arcsin() and arccos() functions lie within the interval $[-1, 1]$. Let ϕ_a and ϕ_b be represented by ϕ, as the expression is symmetrical. From Property 3.4:

$$-1 \leq \frac{1 - {c_2}^2}{c_2} \cot \phi \leq 1.$$

$$\implies -c_2 \leq (1 - {c_2}^2) \cot \phi \leq c_2.$$

Squaring both sides:

$$(1 - {c_2}^2) \cot^2 \phi \geq {c_2}^2.$$

$$\implies {c_2}^2 + {c_2}^2 \cot^2 \phi \geq \cot^2 \phi.$$

$$\implies {c_2}^2 (1 + \cot^2 \phi) \geq \cot^2 \phi.$$

Using $1 + \cot^2 \phi = \csc^2 \phi$:

$${c_2}^2 \csc^2 \phi \geq \cot^2 \phi.$$

Rearranging:

$${c_2}^2 \geq \frac{\cot^2 \phi}{\csc^2 \phi}.$$

Using $\frac{\cot^2 \phi}{\csc^2 \phi} = \cos^2 \phi$:

$${c_2}^2 \geq \cos^2 \phi.$$

Applying this principle to $\arcsin\left(\frac{\cos(\phi_a)}{c_2}\right)$:

$$-1 \leq \frac{\cos(\phi)}{c_2} \leq 1.$$

$$\implies -c_2 \leq \cos(\phi) \leq c_2.$$

Squaring both sides:

$${c_2}^2 \geq \cos^2 \phi.$$

Unfortunately, this means that these expressions are more likely to fail as we near the poles, where $\phi \to 0$ or 90 and $cos^2\phi \to 1$, which is the maximum value c_2 can take.

3.6 Calculating Distances on the Earth

Let us now apply these formulas to calculate a distance along the Earth's surface, namely the shortest distance between Paris Charles de Gaulle Airport (CDG), Paris, France, and Cochin International Airport (COK), Kochi, India, as the crow flies.

Using data from Google Maps[24], we obtain the following latitudinal and longitudinal coordinates to 10 significant figures:

1. Latitude of COK: 10.15352977
2. Longitude of COK: 76.39298831
3. Latitude of CDG: 49.00863258
4. Longitude of CDG: 2.554044990

First we will have to convert this data into the azimuthal and polar angles, as these are given in their latitudinal and longitudinal form. Since the latitude is the angle measured north from the equator, and the polar angle is measured from the pole (z-axis), $\phi = 90° - \text{Latitude}$. However, the longitudes directly correspond to the azimuthal angles.

1. Polar angle of COK = 79.84647023
2. Azimuth of COK = 76.39298831
3. Polar angle of CDG = 40.99136742
4. Azimuth of CDG = 2.55404499

Finally, converting to radians, so we can solve graphically:

1. ϕ_{COK}= 1.393583801609134 rad
2. θ_{COK}= 1.3333091714470386 rad
3. ϕ_{CDG}= 0.7154343263737334 rad
4. θ_{CDG}= 0.044576494319565646 rad

[24]Google Maps, `maps.google.com`.

Now establishing our single-variable equation, using Property 3.5:

$$\theta_{\text{COK}} - \theta_{\text{CDG}} = \arccos\left(\frac{\sqrt{1-{c_2}^2}}{c_2}\cot\phi_{\text{COK}}\right) - \arccos\left(\frac{\sqrt{1-{c_2}^2}}{c_2}\cot\phi_{\text{CDG}}\right)$$

We can graph the above expression, letting $\text{x} = c_2$, and look for the point of intersection:

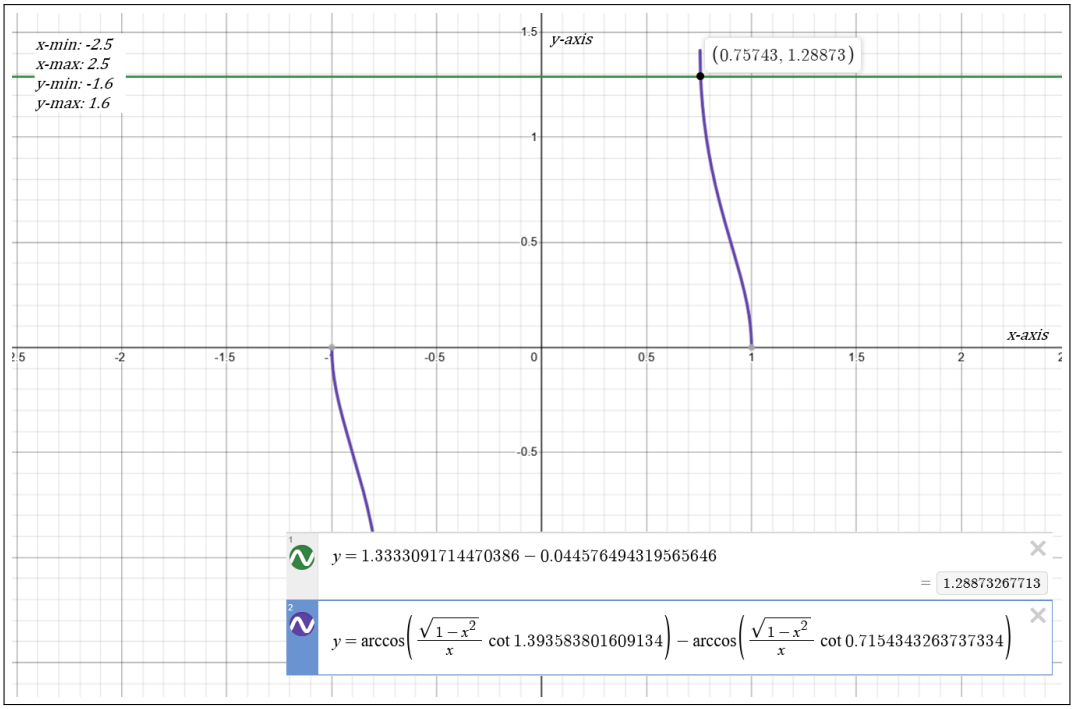

Figure 8: The above system of equations graphed to calculate the point of intersection. [Desmos 2D Graphing Calculator]

Now that we have $c_2 = 0.75743$ and $R = 6378.137km$, as per WGS84, we can insert this into equation 3.4.12 to calculate the distance.

$$L = R\left|\arcsin\left(\frac{\cos(\phi_{COK})}{c_2}\right) - \arcsin\left(\frac{\cos(\phi_{CDG})}{c_2}\right)\right|$$

$$= 6378.137 \times \left|\arcsin\left(\frac{\cos(1.393583801609134)}{0.75743}\right) - \arcsin\left(\frac{\cos(0.7154343263737334)}{0.75743}\right)\right|$$

Solving the above equation using a calculator, the final distance is:

$$L = \mathbf{7989.731690648397\ km}$$

To show that the boundary case is satisfied, let us plug our values for ϕ:

$$\begin{aligned} {c_2}^2 &\geq \cos^2(\phi_{\text{COK}}), & {c_2}^2 &\geq \cos^2(\phi_{\text{CDG}}) \\ &\approx 0.57370 \geq 0.03108, & &\approx 0.57370 \geq 0.5697357479 \end{aligned}$$

As predicted, we are closer to falure in ϕ_{CDG}, which is closer to the North pole.

From a variety of secondary sources, to ensure the data is reliable, we can crosscheck with distances from:

- GoogleMaps: 7980.80 km
- GPS-Coordinates: 7989.67 km
- ProKerala: 7982.82 km
- AirMilesCalculator: 7986 km

Though our value seems to be on the higher side, this can be attributed to our use of a larger spherical model of the Earth, rather than the use of an ellipsoidal model, and we can thus be confident with our answer.

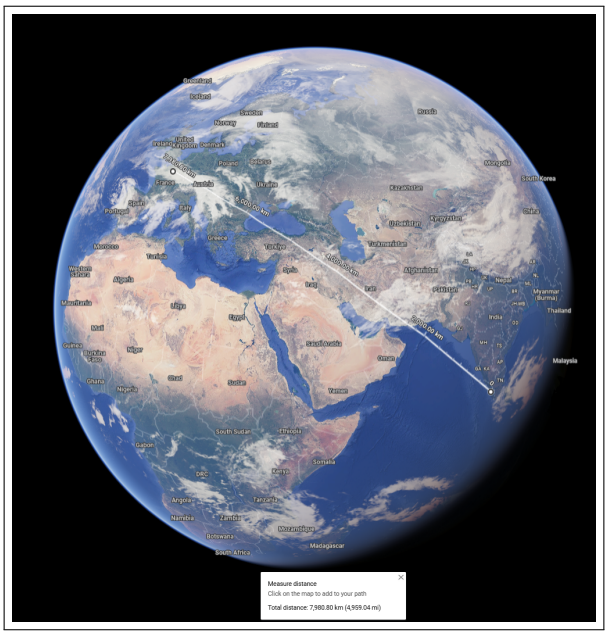

Figure 9: The distance, as the crow flies, from Kochi to Paris, respecting the curvature of the Earth. [Google Maps]

4 Conclusion

In closing, this essay has demonstrated the mathematics behind the Geodesic equations, and its usage in calculating paths of shortest distance along the Earth. The method used, employing the Euler-Lagrange equations, and the First Fundamental Form or Metric Tensor, while certainly being complex and arduous in its application, is widely extendable to a wide variety of surfaces, and extends this essay to generally explain the fundamental concepts of Geodesics.

While we could have used methods such as the Vincenty Formula, which uses a series of iterations to calculate highly accurate distances along the Earth, to see more accurate results and less restrictive boundaries, the expressions would have been limited to the precise geodesy of the Earth, and inextensible to other surfaces or planets.

To further improve the accuracy of the calculations, we can extend our method to ellipsoids, which would involve calculating the metric tensor for a function rotated about the x-axis. Additionally, we could keep further systems of equations rather than heavily simplifying them, to alleviate limitations due to the aforementioned boundary constraints.

The application of geodesics is not limited to the Earth. Geodesics also play an important role in fields such as general relativity, where they define the route light takes following the curvature of spacetime, or in robotics, where they handle tasks from planetary traversal to automobile painting. Therefore, the fact that the derived geodesics equations are applicable on any surface with a given or identifiable metric tensor justifies the essay's prioritization of extensibility in its approach. By deriving three formulas, two as boundary conditions to establish constants of integration and one as the Lagrangian for the path length along the surface, and showcasing an example of the formulas at work, the essay has concisely and successfully answered the research question.

5 References

[1] "14.5: The Chain Rule for Multivariable Functions." Mathematics LibreTexts, 5 Feb. 2025, https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/14\%3A_Differentiation_of_Functions_of_Several_Variables/14.05\%3A_The_Chain_Rule_for_Multivariable_Functions.

[2] "2.7: Basis and Dimension." Mathematics LibreTexts, 17 Sept. 2022, https://math.libretexts.org/Bookshelves/Linear_Algebra/Interactive_Linear_Algebra_(Margalit_and_Rabinoff)/02\%3A_Systems_of_Linear_Equations_Geometry/2.07\%3A_Basis_and_Dimension.

[3] "22.3: Basis Vectors." Mathematics LibreTexts, 17 Sept. 2022, https://math.libretexts.org/Bookshelves/Linear_Algebra/Matrix_Algebra_with_Computational_Applications_(Colbry)/22\%3A_11_In-Class_Assignment_-_Vector_Spaces/22.3\%3A_Basis_Vectors.

[4] "3D Calculator." GeoGebra - the World's Favorite, Free Math Tools Used by over 100 Million Students and Teachers, https://www.geogebra.org/3d.

[5] "Cochin International Airport · Airport Rd, Nedumbassery, Kochi, Kerala 683111, India." Cochin International Airport · Airport Rd, Nedumbassery, Kochi, Kerala 683111, India, https://maps.app.goo.gl/HRpwKrTjozgAEn9T8.

[6] Cheng's Lectures on Relativity & Cosmology. "Derive the geodesic (Lecture #11a of a course on Relativity & Cosmology)." YouTube, January 16, 2022, https://youtube.com/watch?v=KgmdciOko7I.

[7] Dass, H. K. Advanced Engineering Mathematics. S. Chand Publishing, 2007.

[8] Dialect. "An Introduction to Curvilinear Coordinates in Differential Geometry." YouTube, April 13, 2024, https://youtube.com/watch?v=2V__naEkXVY.

[9] ---. "Conceptualizing the Christoffel Symbols: An Adventure in Curvilinear Coordinates." YouTube, October 7, 2023, https://youtube.com/watch?v=TvFvL_sMg4g.

[10] ---. "Demystifying The Metric Tensor in General Relativity." YouTube, October 22, 2021, https://youtube.com/watch?v=Hf-BxbtCg_A.

[11] ---. "The Christoffel Symbols In Riemannian Geometry." YouTube, December 7, 2024, https://youtube.com/watch?v=2992Bqfas_c.

[12] ---. "The Meaning of the Metric Tensor." YouTube, May 7, 2022, https://youtube.com/watch?v=Dn0ZZRVuJcU.

[13] Dr. Trefor Bazett. "The Multi-Variable Chain Rule: Derivatives of Compositions." YouTube, November 14, 2019, https://youtube.com/watch?v=9yCtWfI_Vjg.

[14] ---. "How long is a curve?? The Arclength Formula in 3D." YouTube, October 14, 2019, https://youtube.com/watch?v=80J5s0pic8M.

[15] eigenchris. "Tensor Calculus 0: Introduction." YouTube, March 20, 2018, https://www.youtube.com/watch?v=SmjbpIgVKFs&list=PLJHszsWbB6hpk5h8lSfBkVrpjsqvUGTCx&index=15.

[16] ---. "Tensor Calculus 15: Geodesics and Christoffel Symbols (extrinsic geometry)." YouTube, September 15, 2018, https://youtube.com/watch?v=1CuTNveXJRc.

[17] ---. "Tensor Calculus 16: Geodesic Examples on Plane and Sphere." YouTube, September 20, 2018, https://youtube.com/watch?v=8sVDceI70HM.

[18] Faculty of Khan. "Derivation of the Euler-Lagrange Equation — Calculus of Variations." YouTube, July 17, 2017, https://youtube.com/watch?v=sFqp2lCEvwM.

[19] ---. "The Geodesic Problem on a Sphere — Calculus of Variations." YouTube, December 3, 2018, https://youtube.com/watch?v=1cZC3iMpDxQ.

[20] Fitzpatrick, Richard. "Euclid's Elements". Lulu.com, 2007.

[21] Flammable Maths. "The Leibniz rule for integrals: The Derivation." YouTube, November 28, 2017, https://www.youtube.com/watch?v=Kp7sSp5Kn7o.

[22] Flightradar24. "Live Flight Tracker - Real-Time Flight Tracker Map — Flightradar24." Flightradar24, www.flightradar24.com/data/flights/jl5.

[23] "Fundamental Forms." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource, mathworld.wolfram.com/FundamentalForms.html.

[24] Gauss, Carl Friedrich."General Investigations of Curved Surface". Translated by A. M. Hiltebeitel and J. C. Morehead, Raven Press, 1965.

[25] "Geodesic." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource, 16 Apr. 2002, mathworld.wolfram.com/Geodesic.html.

[26] Good Vibrations with Freeball. "Introduction to Variational Calculus - Deriving the Euler-Lagrange Equation." YouTube, October 22, 2020, https://www.youtube.com/watch?v=6HeQc7CSkZs.

[27] "Graphing Calculator." Desmos, www.desmos.com/calculator.

[28] Group, Vvvv. "Polar, Spherical and Geographic Coordinates." Vvvv - Visual Live-progamming for .NET, 6 Oct. 2015, vvvv.org/blog/2015/polar-spherical-and-geographic/.

[29] Harron, Rob. "MAT-203 : The Leibniz Rule." math.hawaii.edu/rharron/teaching/MAT203/LeibnizRule.pdf.

[30] Khan Academy. "Partial derivatives, introduction." YouTube, May 12, 2016, https://youtube.com/watch?v=AXqhWeUEtQU.

[31] Khan Academy. "Partial derivatives, introduction." YouTube, May 12, 2016, https://www.youtube.com/watch?v=XipB_uEexF0.

[32] Kreyszig, Erwin. "Advanced Engineering Mathematics." John Wiley & Sons, 2010.

[33] Leitão, Henrique, and Joaquim A. Gaspar. "Globes, Rhumb Tables, and the Pre-History of the Mercator Projection." Imago Mundi, vol. 66, no. 2, 2014, pp. 180-195.

[34] Lia, Vas. "Measuring lengths and angles – the first fundamental form." liavas.net, liavas.net/courses/math430/files/Surfaces_part2.pdf.

[35] ---. "The Second Fundamental Form. Geodesics. The Curvature Tensor. The Fundamental Theorem of Surfaces. Manifolds." liavas.net, liavas.net/courses/math430/files/Surfaces_part3.pdf.

[36] "Manifold." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource, mathworld.wolfram.com/Manifold.html.

[37] MathTeacher. "Intro to Space of Continuous and Continuously Differentiable Functions." YouTube, March 4, 2022, https://youtube.com/watch?v=o1SIHCq10EU.

[38] "Mercator, World Map (1569)." Cabinet, www.cabinet.ox.ac.uk/mercator-worldmap-1569-1

[39] "Metric Tensor." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource, 8 Sept. 2014, mathworld.wolfram.com/MetricTensor.html.

[40] Mike, the Mathematician. "Geodesics of the Upper Half Space with Respect for the Hyperbolic Metric." YouTube, November 23, 2024, https://youtube.com/watch?v=xPnk9HJ92CU.

[41] "Paris Charles De Gaulle Airport · 95700 Roissy-en-France, France." Paris Charles De Gaulle Airport · 95700 Roissy-en-France, France, maps.app.goo.gl/tKw2n3tipzfAbgJG6.

[42] "Partial Derivatives." Math is Fun, www.mathsisfun.com/calculus/derivatives-partial.html.

[43] "Partial Derivative." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource, 25 Nov. 2008, mathworld.wolfram.com/PartialDerivative.html.

[44] Spivak, Michael. "A Comprehensive Introduction to Differential Geometry" Publish or Perish, 1999.

[45] Tawfeek, Andrew. "Obtaining the Shortest Distance Between Two Points on a Surface." ResearchGate, Dec. www.researchgate.net/publication/311647003_Obtaining_the_Shortest_Distance_Between_Two_Points_on_a_Surface.

[46] Veblen, Oswald, and J. H. Whitehead. The Foundations of Differential Geometry. 1953.

[47] "Vector Space." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource, 10 Jan. 2009, mathworld.wolfram.com/VectorSpace.html.

[48] Weinstock, Robert. "Calculus of Variations - With Applications to Physics and Engineering (1974)", Read Books, 2008.

[49] Wolfram—Alpha: Making the World's Knowledge Computable, www.wolframalpha.com/.

[50] "World Geodetic System 1984 (WGS 84)". Office of Geomatics, National Geospatial-Intelligence Agency.